PROOF THAT $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ using the Well-Ordering Principle of $\mathbb{Z}$.

To Prove: For all integers $n \geq 1$, $1+2+\cdots+n = \frac{n(n+1)}{2}$.

Proof: Suppose, by way of Contradiction, that there exists an integer n_0 such that $n_0 \geq 1$ and
$$1+2+\cdots+n_0 \neq \frac{(n_0)(n_0+1)}{2}.$$

Let $S = \{$all integers x such that $x \geq 1$ and $1+2+\cdots+x \neq \frac{x(x+1)}{2}.\}$

By supposition, $n_0 \geq 1$ and $1+2+\cdots+n_0 \neq \frac{n_0(n_0+1)}{2}$.

$\therefore$ Integer n_0 is in set S, and so set S is not empty.

$\therefore$ Condition 1 of the Well-Ordering Principle is satisfied.

By definition of set S, for every x in S, $x \geq 1$.

$\therefore$ Condition 2 of the Well-Ordering Principle is satisfied.

$\therefore$ By the Well-Ordering Principle, set S has a least element, m.

$\therefore m \geq 1$ and $1+2+\cdots+m \neq \frac{m(m+1)}{2}$ and m is the least positive integer with both of these properties.

[Note that this means that, if x is an integer such that $x \geq 1$ and x is NOT in set S, then $1+2+\cdots+x = \frac{x(x+1)}{2}$, or otherwise x would be in set S.]

Now, $m-1 < m$.

$\therefore$ $m-1$ is not in set S because m is the least element in set S.

[We need to show that $m-1 \geq 1$, that is, that $m \geq 2$ which follows if we show that $m \neq 1$]

Let $n=1$. Then, $1+2+\cdots+n = 1$ and $\frac{n(n+1)}{2} = \frac{(1)(2)}{2} = 1$.

$\therefore$ For $n=1$, $1+2+\cdots+n = \frac{n(n+1)}{2}$. Thus, $n=1$ is not in set S, so $m \neq n = 1$, because $1+2+\cdots+m \neq \frac{m(m+1)}{2}$.

$\therefore m \geq 2$.

$\therefore m-1 \geq 1$

Since $m-1 \geq 1$ and $m-1$ is NOT in set S,

$\therefore 1+2+\cdots+(m-1) = \frac{(m-1)(m)}{2}$. [See NOTE ABOVE]

$\therefore \left(1+2+\cdots+(m-1)\right) + m = \frac{(m-1)(m)}{2} + m$, by substitution,

$\therefore 1+2+\cdots+(m-1)+m = \frac{(m-1)m+2m}{2} = \frac{(m-1+2)m}{2}$.

$\therefore 1+2+\cdots+m = \frac{m(m+1)}{2}$, which contradicts the fact that $1+2+\cdots+m \neq \frac{m(m+1)}{2}$.

$\therefore$ By proof-by-contradiction, for all integers $n \geq 1$,

$$1+2+\cdots+n = \frac{n(n+1)}{2}.$$

QED